SPIRAL WAVES AND THE DYNAMICAL SYSTEM APPROACH

PATRICK BOILY

ABSTRACT. Spirals are common in Nature: the snail's shell and the ordering of seeds in the sunflower are amongst the most widely-known occurrences. While these are static, dynamic spirals can also be observed in excitable systems such as heart tissue, retina, certain chemical reactions, slime mold aggregates, flame fronts, etc. The images associated with these spirals are often breathtaking, but spirals have also been linked to cardiac arrhythmias, a potentially fatal heart ailment.

In the literature, very specific models depending on the excitable system of interest are used to explain the observed behaviour of spirals (such as anchoring or drifting). Barkley [4] first noticed that the Euclidean symmetry of these models, and not the model itself, is responsible for the observed behaviour. But in experiments, the physical domain is never Euclidean. The heart, for instance, is finite, anisotropic and littered with inhomogeneities. To capture this loss of symmetry, LeBlanc and Wulff [34, 35] introduced forced Euclidean symmetry-breaking (FESB) in the analysis.

To accurately model the physical situation, two basic types of symmetry-breaking perturbations are used: translational symmetry-breaking (TSB) and rotational symmetry-breaking (RSB) terms. In this paper, we provide an overview of currently know results about spiral wave dynamics under FESB.

1. INTRODUCTION

The spiral is an integral part of Nature: it can be seen in a snail's shell, in the layout of a sunflower's seeds and in the path of a falcon on a hunt, to name but a few. These particular instances are fixed in space, but spirals can also evolve in time: hurricanes and galaxies are common examples that come to mind. It is, however, rather arduous to conduct experiments on the latter physical objects, for obvious reasons.

On a smaller scale, where experiments are easier to control, spirals in evolution have also been observed in *excitable media* such as heart tissue, slime-mold aggregates, the retina or certain chemical reactions (such as the famed Belousov-Zhabotinsky (BZ) reaction). In these systems, waves propagate by 'exciting' a 'cell', which in turn 'excites' some of its neighbours before falling into a 'refractory' or 'unexcitable' state, followed by a 'resting' state, ready to be 'excited' should the wave come its way again.

These systems give rise to beautiful images (as can be attested to in figure 1). While this in itself might yield enough interest to study them, there is also (at least) one serious reason to do so: spiral waves have been linked to *cardiac arrhythmias*, *i.e.* to disruptions of the heart's normal electrical cycle [54, 55]. Most arrhythmias are harmless but if they are 're-entrant in nature and [...] occur [in the ventricles] because of the spatial distribution of cardiac tissue [30, p. 401]', they can seriously hamper the pumping mechanism of the heart and so lead to death. As a result, a full understanding of spiral wave dynamics in these media becomes imperative.
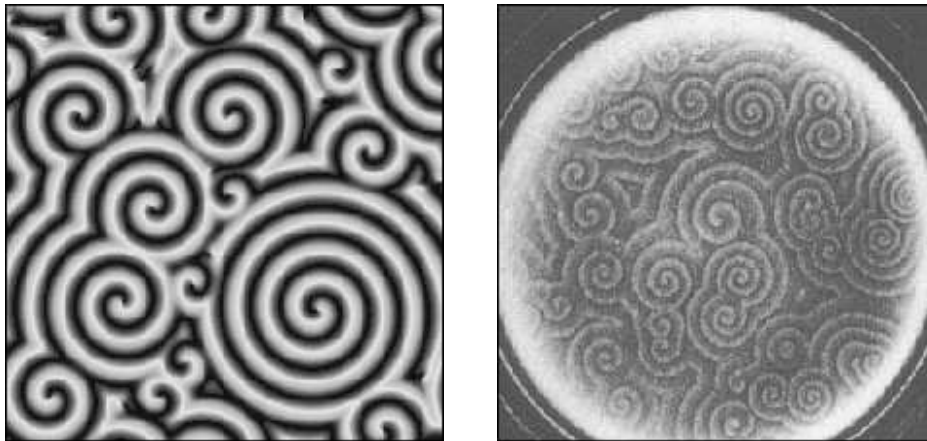
FIGURE 1. Spirals in excitable media. On the left, spirals in a solution of the Ginzburg-Landau equation [26], on the right, spirals in colonies of the slime-mold aggregate *Dictyostelium discoideum* [3].

2. HISTORICAL PERSPECTIVE

Numerous experiments and simulations have been performed with excitable media, see for instance [4,5,7,17,36] (a selected bibliography can be found in appendix A). The various 'spiral' motions that are observed are classified according to their *tip path*, an arbitrary point on the wave front that is followed in time, as can be seen, for instance, in figure 2. Some of the standard possibilities are shown in figure 3.

In the literature (see [6,41,53] for instance), specific systems of partial differential equations (PDE) have been used to attempt to explain the observed phenomena: for instance, the FitzHugh-Nagumo equations

$$u_t = \tfrac{1}{\varsigma}(u - \tfrac{1}{3}u^3 - v) + \Delta u$$
$$v_t = \varsigma(u + \beta - \gamma v)$$

of cardiology, where ς, β and γ are model parameters, u represents an electric potential and v a measure of permeability, or the Oregonator

$$u_t = \tfrac{1}{\varsigma}(u - u^2 - fv\tfrac{u-q}{u+q}) + \Delta u$$
$$v_t = u - v + D_v \Delta v$$

for the Belousov-Zhabotinsky reaction, where f, ς and q (small) are the model parameters, D_v is a diffusion coefficient and u and v represent the concentrations of certain chemical reactants.

These two systems of partial differential equations are instances of a general class of PDE, given by

$$(2.1) \qquad\qquad u_t(x,t) = f(u(x,t)) + D\Delta u(x,t),$$

where $x \in \mathbb{R}^2$, $u : \mathbb{R}^2 \times \mathbb{R}_0^+ \to \mathbb{R}^m$ is bounded and uniformly continuous, D is an $m \times m$ diagonal matrix and $f : \mathbb{R}^m \to \mathbb{R}^m$ is some sufficiently smooth function. General systems of the form (2.1) are called *reaction-diffusion system* (RDS).[1]

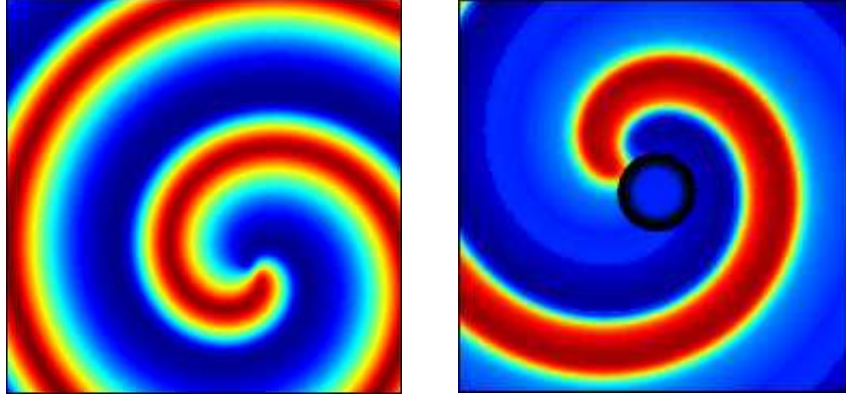[1]Winfree provides a very complete survey of their use as models [53].

FIGURE 2. On the left, an isolated spiral in a RDS; the excited wave front is shown in red. On the right, the corresponding tip path.
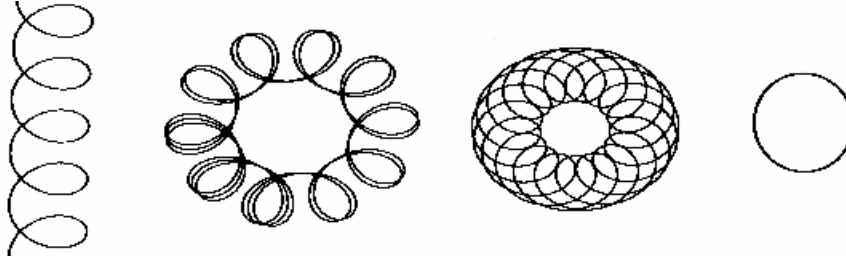

FIGURE 3. Possible paths of the spiral wave tip. From left to right: linear drifting, outer epicycle motion, inner epicycle motion and rigid rotation [35].

Four types of solutions of (2.1) are of particular interest in the context of spiral wave dynamics:

Rotating waves (RW) are rigidly rotating periodic solutions that are fixed in a *co-rotating* frame of reference (*i.e.* the frame rotates uniformly with the same frequency as the solution). In physical and numerical experiments, the tip path is circular. RW are sometimes called *vortices* or *rotors* in the literature [53, 55].

Traveling waves (TW) are linearly propagating solutions that are fixed in a *co-translating* frame of reference (*i.e.* the frame translates linearly and uniformly with the solution). In experiments, the tip path of such a solution is a line. Strictly speaking, TW (or *retracting tip waves* [6]) are not spiral waves as they do not have a rotating component.

Modulated rotating waves (MRW) are two-frequency quasi-periodic solutions that are periodic in a co-rotating frame of reference that rotates uniformly with one of the frequencies of the solution. The tip path of such a solution is a closed epicycle when the ratio of the frequencies is rational; otherwise the tip path densely fills a ring over time, with an epicycle-like motion. In the literature, MRW are sometimes called *meandering waves*.
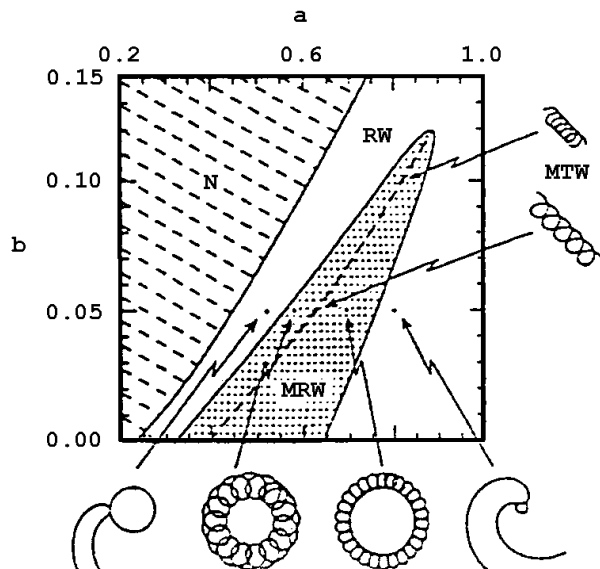
FIGURE 4.   Bifurcation diagram of spiral wave dynamics in (2.2) [5].

Modulated traveling waves (MTW) are rotating solutions, superimposed
with a linearly propagating motion, that are periodic in a co-translating
frame of reference that travels uniformly with the linear component of the
solution. The tip path of such a solution is a helix-shaped two-dimensional
curve.

As an example of a RDS in which these occur, consider Barkley's system:

$$(2.2) \qquad \begin{aligned} u_t &= \tfrac{1}{\varsigma}u(1-u)\left(u - \tfrac{v+b}{a}\right) + \Delta u \\ v_t &= u - v, \end{aligned}$$

where a, b and ς are system parameters with ε small [5]. Figure 4 shows a bifurcation
diagram of the spiral dynamics of (2.2) for $\varepsilon = \frac{1}{50}$. There are three regions of interest
labeled N, RW and MRW. Note that this last region is divided in two sub-regions by a
curve labeled MTW.

In N, no wave propagation is observed; in RW, observed solutions are RW and
in MRW, observed solutions are MRW, with petality[2] determined by the side of the
curve MTW on which the parameters fall. The intersection of this curve with the
boundary of MRW is a point that deserves special consideration: in every one of its
neighbourhoods, the three basic types of spiral behaviours can be seen.

An *a priori* surprising feature of reaction-diffusion systems is that figure 4 is a
generic bifurcation diagram: most experimental results are strikingly similar [6,53];
this suggests they are in fact a consequence of excitable media and their geometry,
and not of the particular models that aim to describe the dynamics [4, 21, 34, 35].[3]
This is where the dynamical system approach enters the picture.

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[2]The orientation of the spiral "petals".

[3]It should be noted that reaction-diffusion systems are not the sole models of excitable media,
nor were they the first: Wiener and Rosenblueth originally defined and modeled excitable media
using cellular automata [52].

3. Equivariant Vector Fields and Reaction-Diffusion Systems

Let Γ be a group acting linearly on a vector space X. A function $f : X \to X$ is $\Gamma-$*equivariant* if it commutes with the action of Γ, *i.e.*

$$\gamma \cdot f(x) = f(\gamma \cdot x), \quad \forall \gamma \in \Gamma, x \in X.$$

Equivariant vector fields (with compact Γ) have been studied by many authors: notable amongst them are Golubitsky and Schaeffer [23], Golubitsky, Stewart and Schaeffer [24] and Vanderbauwhede [50,51]. The main feature of these $\Gamma-$equivariant vector fields is that whenever $x(t)$ is a solution of $\dot{x} = f(x)$, so is $\gamma x(t)$, for all $\gamma \in \Gamma$.

The *special Euclidean group* $\mathbb{SE}(2) = \mathbb{C}\dot{+}\mathbb{S}^1$ is a non-compact subset of all the distance-preserving transformations of the plane, with multiplication defined by

$$(3.1) \quad (p_1, \varphi_1) \cdot (p_2, \varphi_2) = (e^{i\varphi_1} p_2 + p_1, \varphi_1 + \varphi_2), \quad \forall (p_1, \varphi_1), (p_2, \varphi_2) \in \mathbb{SE}(2).$$

It acts on the space of bounded uniformly continuous functions from $\mathbb{R}^2$ to $\mathbb{R}^m$, which we will denote by $\mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$, according to

$$(3.2) \quad (\gamma \cdot v)(x) = ((p, \varphi) \cdot v)(x) = v(R_{-\varphi}(x - p)), \quad \forall (p, \varphi) \in \mathbb{SE}(2),$$

where R_θ represents a rotation by angle θ around the origin. Reaction-diffusion systems on $\mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$ are $\mathbb{SE}(2)-$equivariant under the action of (3.2), but that action is not smooth over $\mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$: the problem arises with rotations, as a small shift in θ produces a large displacement at far distances [56]. However, there is a closed set $\mathrm{BC}_e(\mathbb{R}^2, \mathbb{R}^m) \subsetneq \mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$ over which (3.2) is smooth [56].

3.1. Abstract Differential Equations.
In order to determine $\mathrm{BC}_e(\mathbb{R}^2, \mathbb{R}^m)$, Wulff uses the following RDS paradigm [54]. Consider

$$(3.3) \quad u_t(x, t) = \tilde{D}\Delta u(x, t) + f(u(x, t), \varsigma),$$

where $x \in \mathbb{R}^2$, $u : \mathbb{R}^2 \times \mathbb{R}_0^+ \to \mathbb{R}^m$, $\tilde{D} \geq 0$ is a diagonal matrix, $\varsigma \in \mathbb{R}^M$, Δ is the Laplacian and f is C^{k+2} for some $0 \leq k \leq \infty$. If $\det \tilde{D} \neq 0$, let $Y = \mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$ be the Banach space of *uniformly continuous, bounded functions from $\mathbb{R}^2$ to $\mathbb{R}^m$*. Otherwise, as long as f satisfies some additional growth conditions, the choice $Y = L^2(\mathbb{R}^2, \mathbb{R}^m)$ can also be used, with slight variations (see [27,44] for details).[4]

The *semi-linear differential equation on Y associated to (3.3)* is the abstract differential equation

$$(3.4) \quad \frac{du}{dt} = -Au + F(u, t, \varsigma),$$

where $F(u, t, \varsigma) = f(u(\cdot, t), \varsigma)$ and

$$(3.5) \quad A = \mathrm{diag}(-d_1\Delta, \ldots, -d_m\Delta).$$

Solutions of (3.4) are in one-to-one correspondence with solutions of (3.3).

In the remainder of this section, we assume the reader is familiar with basic definitions and results from operator theory and abstract differential equations (see [2, 27, 56] for details and definitions).

Proposition 3.1. [56] *Let A be given by (3.5), $Y = \mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$, $\alpha \in (\frac{1}{2}, 1)$ and $A_1 = \mathrm{id}_Y - \tilde{D}\Delta$, where $\tilde{D}$ is as in (3.3). Then A is sectorial in Y^α and $\frac{\partial}{\partial x_1} A_1^{-\alpha}, \frac{\partial}{\partial x_2} A_1^{-\alpha}$ are bounded on Y.*

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[4]However, physical considerations demand that spiral waves in an infinitely extended medium be located in $\mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$ [56].

3.2. **Existence and Uniqueness of Solutions.** That (3.4) has classical solutions is shown by the following theorem.

Theorem 3.2. [27, 56] *Let Y, A, A_1 and α be as in proposition (3.1), U be a subset of $Y^\alpha \times \mathbb{R} \times \mathbb{R}^M$ and F be as in (3.4), locally Lipschitz in its first variable and continuous in the remaining variables. Then, for any $(u_0, t_0, \varsigma) \in U$, (3.4) has a unique classical $C^{k+2}-$solution $u(t; u_0, t_0, \varsigma)$ on $[t_0, t_1]$, where $t_1 = t_1(u_0, t_0, \varsigma) > t_0$ and $0 \le k \le \infty$ is the smoothness of the nonlinearity f in (3.3).*

3.3. $\mathbb{SE}(2)-$**Equivariance of Solutions.** Wulff then shows that these solutions have a $\mathbb{SE}(2)-$equivariant structure. A *smooth local semi-flow* $\{\Phi_t\}_{t \ge 0}$ on a Banach space X is a smooth family of operators satisfying $\Phi_0 = \mathrm{id}_X$ and

$$(3.6) \qquad \Phi_{t+s} = \Phi_{t,\varsigma} \circ \Phi_s = \Phi_s \circ \Phi_t \quad \text{for all } s, t \ge 0.$$

If furthermore $\Phi_t x \to x$ as $t \to 0^+$ for all $x \in X$, then $\{\Phi_t\}_{t \ge 0}$ is a C^0-*semi-group* on X. A *smooth semi-group* on X is a C^0-semi-group for which the map $\Psi_x : (0, \infty) \to X$ defined by $\Psi_x(t) = \Phi_t(x)$ is smooth for all $x \in X$.

The *infinitesimal generator* L_T of a smooth semi-group $\{T_t\}_{t \ge 0}$ on a Banach space X is

$$(3.7) \qquad L_T x = \lim_{t \to 0^+} \frac{1}{t} \left(\Phi_t x - x \right),$$

whenever the limit exists.

The *special Euclidean group* $\mathbb{SE}(2)$ plays an important role in the theory of spiral waves. For now, we assume it is parameterized as $\mathbb{SE}(2) = \mathbb{SO}(2) \dot{+} \mathbb{R}^2$, with multiplication given by

$$(3.8) \qquad (R_1, S_1) \cdot (R_2, S_2) = (R_1 R_2, S_1 + R_2 S_2).$$

The *standard* $\mathbb{SE}(2)-$*action on* $\mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$ is

$$(3.9) \qquad (R, S)\,(v(x,t)) = v\left(R^{-1}(x - S), t \right).$$

Note that $\mathbb{SE}(2)$ is generated by the families $\{S_\mu^1\}$, $\{S_\nu^2\}$ and $\{R_\theta\}$, where

$$S_\mu^1 = \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \quad S_\nu^2 = \begin{pmatrix} 0 \\ \nu \end{pmatrix} \quad \text{and} \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

satisfy (3.6). Of these, only $\{S_\mu^1\}_{\mu \ge 0}$ and $\{S_\nu^2\}_{\nu \ge 0}$ are smooth semi-groups on $\mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$; their respective infinitesimal generators are

$$L_{S^1} = -\frac{\partial}{\partial x_1} \quad \text{and} \quad L_{S^2} = -\frac{\partial}{\partial x_2}.$$

On the other hand, $\{R_\theta\}$ is not a smooth semi-group (see [56, lemma 2.14]); the action of $\mathbb{SE}(2)$ on $\mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$ is not even continuous. Thus, $\mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$ is not a suitable space over which to define (3.3).

This obstacle is overcome as follows. The formal evaluation of (3.7) yields

$$L_R = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}.$$

Now let $\tilde{Y} = \mathrm{BC}_e(\mathbb{R}^2, \mathbb{R}^m)$ be the topological closure of $\mathrm{domain}(L_R)$ in $\mathrm{BC}_u(\mathbb{R}^2, \mathbb{R}^m)$. The reaction-diffusion system (3.3) is well-posed on $\tilde{Y}$, and proposition (3.1) and theorem 3.2 still hold after substituting $\tilde{Y}$ for Y [56]. Furthermore, (3.9) is continuous on $\tilde{Y}$ and the following result holds.

Theorem 3.3. [18, 49, 56] *The semi-flow $\Phi_{t,\varsigma}$ generated by (3.3) commutes with the restricted action (3.9) of $\mathbb{SE}(2)$ over $\mathrm{BC}_e(\mathbb{R}^2, \mathbb{R}^m)$.*

In particular, the other results of her thesis hold as long as $\Phi_{t,\varsigma}$ is a smooth $\mathbb{SE}(2)-$equivariant semi-flow (not necessarily generated by a RDS) on some suitable Banach space.

4. Barkley's Insight

Barkley was the first to realize that the Euclidean symmetry discussed in the previous section (as opposed to the specifics of a given model) could explain most of the spiral wave dynamics observed in experiments and simulations, and succinctly presented in figure 4 [4,5]. His key observation rests on the fact that for any reaction-diffusion system, the linearization at a RW at the onset of a Hopf bifurcation (hence at the boundary of MRW) has five isolated leading eigenvalues on the imaginary axis: $\lambda_R = 0$ (due to rotational symmetry), $\lambda_T = \pm i\omega$ (due to translational symmetry) and $\lambda_B = \pm i\beta_0$ (responsible for the Hopf bifurcation from RW to MRW or *vice-versa*).

The pairs of complex conjugate eigenvalues λ_T and λ_B can be made to coincide by varying two or more system parameters. The corresponding (interesting) codimension-two point is then found to lie precisely at the intersection of MTW and the boundary of MRW.

4.1. **Linear Stability Analysis at a RW.** In his ground-breaking paper [4], Barkley considers the Oregonator-like system

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + \frac{1}{\varepsilon}u_1(1-u_1)\left(u_1 - \frac{u_2+b}{a}\right)$$

$$\frac{\partial u_2}{\partial t} = \delta\Delta u_2 + u_1 - u_2,$$

where a, b, ε are parameters with ε small and $\delta \in [0,1]$ constant. In vector form, this system may be written as

$$\frac{\partial u}{\partial t} = \hat{\delta}\Delta u + f(u), \tag{4.1}$$

where $u = (u_1, u_2)^\top$, $\hat{\delta} = \mathrm{diag}(1,\delta)$ and f contains the remaining terms. The boundary conditions $\partial_r u = 0$ is taken on a circle of radius $R > 0$. With this set-up, (4.1) is $\mathbb{SE}(2)-$equivariant under the action of (3.9), on some suitable Banach function space, as $R \to \infty$.

To find RW solutions of (4.1), *i.e.* solutions for which $(\partial_t + \omega\partial_\theta)u \equiv 0$ for some speed of rotation ω, it suffices to solve the eigenvalue problem

$$F(u) = 0$$
$$DF(u)\tilde{u} = \lambda\tilde{u}, \tag{4.2}$$

where $F(u) = \hat{\delta}\Delta u + \omega\partial_\theta u + f(u)$ and $DF(u) = \hat{\delta}\Delta + \omega\partial_\theta + Df(u)$. Any λ solving (4.2) corresponds to an eigenvalue of the linearization of (4.1) at the RW solution.

Using fast and efficient numerical methods, Barkley shows that three of the five leading eigenvalues[5] lie on the imaginary axis. Indeed, the rotational symmetry of (4.1) forces $\lambda_R = 0$ (with corresponding eigenmode $\tilde{u}_R = \partial_\theta u$, where u is the

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[5]Eigenvalues with largest real part.

spiral solution of the first equation in (4.2)); the translational symmetry of (4.1) imposes $\lambda_T = \pm i\omega$ (with corresponding eigenmode $\tilde{u}_T = \partial_x u \pm i\partial_y u$, where u is as above). Note that this holds in spite of the fact that the boundary condition breaks the Euclidean symmetry: for sufficiently large domain, the real part of λ_T is numerically indistinguishable from zero.[6]

There is a last pair of complex conjugate leading eigenvalues $\lambda_B = \alpha(a) \pm i\beta(a)$ that crosses the imaginary axis for some prescribed $a = a^*$, leading to a Hopf bifurcation or 'spiral wave instability', in which MRW are observed.

All the remaining eigenvalues have negative real part and so do not affect spiral dynamics. As a result, the five leading eigenvalues are isolated in the spectrum and so any pair (u, ω_{rot}) that solves (4.2) is *not* part of a continuum of solutions with continuously varying shapes or speed of rotation.

These results are in fact model-independent; as long as the reaction-diffusion equations governing the field u are $\mathbb{SE}(2)-$equivariant, the five leading eigenvalues will have the above properties.

4.2. **The Ad-Hoc Model.** Based on these observation, Barkley [6] constructed an *ad hoc* $5-$dimensional system of ordinary differential equations (ODE) which replicates the above resonant Hopf bifurcation:

$$\dot{p} = v$$
$$(4.3) \qquad \dot{v} = v\left[f(|v|^2, w^2) + iwh(|v|^2, w^2)\right]$$
$$\dot{w} = wg(|v|^2, w^2)$$

where $p, v \in \mathbb{C}$, $w \in \mathbb{R}$ and

$$f(\xi, \zeta) = -\frac{1}{4} + \alpha_1\xi + \alpha_2\zeta - \xi^2, \quad g(\xi, \zeta) = \xi - \zeta - 1, \quad \text{and} \quad h(\xi, \zeta) = \gamma_0$$

for some $\gamma_0 \in \mathbb{R}$. The variable p represents the position of the spiral tip, while v is its linear velocity and $\gamma_0 w$ its instantaneous rotational frequency rate. This system has RW solutions that undergo a Hopf bifurcation to MRW solutions and it also has a codimension-two resonant Hopf point. Furthermore, it is equivariant under the distance-preserving planar transformations generated by

$$R_\gamma \begin{pmatrix} p \\ v \\ w \end{pmatrix} = \begin{pmatrix} e^{i\gamma}p \\ e^{i\gamma}v \\ w \end{pmatrix} \quad \text{and} \quad T_{\alpha,\beta} \begin{pmatrix} p \\ v \\ w \end{pmatrix} = \begin{pmatrix} p + \alpha + i\beta \\ v \\ w \end{pmatrix},$$

where R_γ and $T_{\alpha,\beta}$ represent respectively a rotation by angle γ around the origin and a translation by the vector $\alpha + i\beta$.

Note the absence of p in the right-hand side of (4.3) as position plays no role in Euclidean systems. Figure 5 shows the bifurcation diagram of (4.3) for $\gamma_0 = 5.6$. The similarities with figure 4 are readily apparent, in particular when it comes to the presence of a codimension-two organizing center around which the three types of spirals can be found.

5. THE DYNAMICAL SYSTEM APPROACH

Then, in what has been hailed a "major mathematical work on spirals [21]", Wulff [56] rigourously proved that the resonant unbounded growth observed by many authors (such as [4,6]) does indeed occur near the codimension-two point.

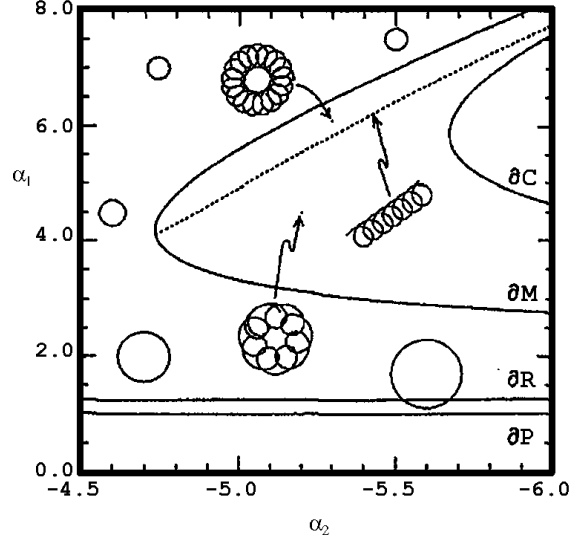[6]Barkley provides some very strong estimates to that effect.

FIGURE 5. Bifurcation diagram of (4.3) [6].

The following result, the center manifold reduction theorem of Sandstede, Scheel and Wulff [19, 43–45], remains, in the author's opinion, both the most technical general results on spiral wave dynamics and its most fruitful ally in applications. It is an extension to non-compact symmetry group of Krupa's [31] center bundle construction for relative equilibria and periodic solutions.

5.1. **The Center Manifold Reduction Theorem (CMRT).** The CMRT helps provide a rigorous link between spiral solutions of (3.3) and Barkley's *ad hoc* model (4.3). Set $\mathfrak{se}(2) = \mathfrak{so}(2) \times \mathbb{R}^2$, where $\mathfrak{so}(2)$ is the Lie algebra of $\mathbb{SO}(2)$, consisting of the 2×2 anti-symmetric matrices on $\mathbb{R}$. As a one-dimensional vector space,

$$\mathfrak{so}(2) = \mathrm{Span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} = \mathrm{Span}\{J_2\}.$$

Define $\exp_{\mathfrak{so}(2)} : \mathfrak{so}(2) \to \mathbb{SO}(2)$ by

$$\exp_{\mathfrak{so}(2)}(bJ_2) = \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix}.$$

Then, $\mathfrak{se}(2)$ is the Lie algebra of $\mathbb{SE}(2)$, when endowed with commutator and exponential maps as defined by (5.1) and (5.2) below. Let I_2 be the 2×2 identity matrix. Then

(5.1) $\qquad [(r_1, s_1), (r_2, s_2)] = (r_1 r_2 - r_2 r_1, r_1 s_2 - r_2 s_1)$

(5.2) $\qquad \exp((r, s)t) = (\exp_{\mathfrak{so}(2)}(rt), r^{-1}(\exp_{\mathfrak{so}(2)}(rt) - I_2)s),$

for $t \in \mathbb{R}$, $r_j \in \mathfrak{so}(2)$ and $s_j \in \mathbb{R}^2$, $j = \varnothing, 1, 2$ [19, 44].

Now, consider a RDS of the form (3.3). For a fixed ς, a *relative equilibrium* of (3.3) is a solution $u(x, t)$ whose time orbit, or semi-flow orbit, is contained in its group orbit under $\mathbb{SE}(2)$. More precisely, it satisfies

(5.3) $\qquad\qquad u(x, t) = \exp((r, s)t)u(x, 0)$

for some $(r, s) \in \mathfrak{se}(2)$. The *isotropy subgroup* of such solutions is

$$\Sigma_u = \{\sigma \in \mathbb{SE}(2) : \sigma u(x, t) = u(x, t)\}.$$

If $\Sigma_u \cong \mathbb{Z}_\ell$, then $u(x, t)$ is an $\ell-$*armed spiral*.

According to the definitions of section 2, RW are relative equilibria of (3.3) with no translation component; after a change of coordinates bringing the center of rotation to the origin, (5.3) becomes

$$(5.4) \qquad\qquad u_*(x, t) = (\exp_{\mathfrak{so}(2)}(r_* t), 0) u_*(x, 0)$$

for some non-trivial $r_* \in \mathfrak{so}(2)$.

Similarly, a *relative periodic solution* of (3.3) is a solution

$$u(x, t) = \exp((r, s) t) w(x, t),$$

for some $(r, s) \in \mathfrak{se}(2)$, where w is a non-constant $T-$periodic function in t. According to the definitions of section 2, MRW are relative periodic solutions of (3.3) with no translation component; after a change of coordinates bringing the center of rotation to the origin, (5.3) becomes

$$(5.5) \qquad\qquad u^*(x, t) = (\exp_{\mathfrak{so}(2)}(r^* t), 0) w(x, t)$$

for some non-trivial $r^* \in \mathfrak{so}(2)$ and a non-constant $T-$periodic function w.[7] The isotropy subgroup of u^* is defined as for RW, and the interpretation is identical.

Standard center bundle results (such as those presented by Krupa [31]) cannot be applied to (3.3) because $\mathbb{SE}(2)$ is not compact (a small rotation at the origin will produce a large displacement away from the origin). However, it can be shown that relative equilibria and relative periodic solutions are members of $\mathrm{BC}_e(\mathbb{R}^2, \mathbb{R}^m)$, and so, as was seen in section 3, that the action of $\mathbb{SE}(2)$ defined by (3.9) is continuous on RW and MRW.

The two hypotheses that follow allow the resolution of some technical difficulties within the CMRT.

Hypothesis 1. *For the parameter ς_* [resp. ς^*], assume u_* [resp. u^*] is a $1-$armed (normally hyperbolic) RW [resp. MRW] of (3.3) with $0 \neq r_*$ as in (5.4) [resp. with $0 \neq r^*$ and w as in (5.5)]. If $\tilde{D}$ is singular, assume further that u_* [resp. u^*] is $k + 2-$times uniformly continuously differentiable.*

Scheel [46] has shown that such rotating wave solutions can arise from Hopf bifurcations in a large class of planar reaction-diffusion equations.

Hypothesis 2. *Assume that $\{\mu : |\mu| \geq 1\}$ is a spectral set for the linearization $\exp_{\mathfrak{so}(2)}(-r_*) D\Phi_{1,\varsigma_*}(u_*)$ [resp. $\exp_{\mathfrak{so}(2)}(-r^*) D\Phi_{1,\varsigma^*}(u^*)$] and that*

$$\dim(\mathrm{range}(P_*)) = 3,$$

[resp. $\dim(\mathrm{range}(P^*)) = 5$], where P_* [resp. P^*] is the spectral projection associated to u_* [resp. u^*].

That this second hypothesis *can* hold has been verified numerically by Barkley [4] (see section 4). It should be noted, however, that Scheel [46] has also shown that this hypothesis fails to hold for a large class of asymptotically Archimedean spiral waves.

---

[7]While we focus mainly on RW and MRW, the theorems of this section can easily be adapted for TW and MTW.

The following result establishes the existence of an invariant center manifold that is contained in an $\mathbb{SE}(2)-$invariant neighbourhood of the group orbit of u_* [resp. u^*].

Theorem 5.1. ([44], theorem 4, p. 142) *For any ς close enough to ς_* [resp. ς^*], there exists an $\mathbb{SE}(2)-$invariant, locally semi-flow-invariant manifold M_ς^{cu}. Both M_ς^{cu} and the action of $\mathbb{SE}(2)$ on M_ς^{cu} are C^{k+1} and depend $C^{k+1}-$smoothly on ς. Furthermore, M_ς^{cu} contains all solutions which stay close to the group orbit of u_* [resp. u^*] for all negative times. Finally, M_ς^{cu} is locally exponentially attracting. Furthermore, the manifold M_ς^{cu} is diffeomorphic to the bundle $V_* = \mathbb{C} \times \mathbb{S}^1$ [resp. $V^* = \mathbb{C} \times \mathbb{T}$].*

5.2. **The CMRT and its Applications.** In [21], Golubitsky, LeBlanc and Melbourne describe the structure of the equations on M_ς^{cu} assuming the spiral waves have trivial isotropy subgroup. Generally, the essential dynamics for Hopf bifurcation from $\ell-$armed spirals are analyzed *via* a $5-$dimensional system of ODE on the center bundle $\mathbb{SE}(2) \times \mathbb{C}$ describes. For $\ell = 1$, the general system reduces to the center bundle equations

(5.6)
$$\dot{p} = e^{i\varphi} F^p(q, \overline{q})$$
$$\dot{\varphi} = F^\varphi(q, \overline{q})$$
$$\dot{q} = F^q(q, \overline{q}),$$

where $p, q \in \mathbb{C}$, $\varphi \in \mathbb{S}^1$, $F^\varphi(0) = \omega_{\mathrm{rot}} \in \mathbb{R}$, $F^q(0) = 0$ and $DF^q(0) = i\omega_{\mathrm{per}}$ is purely imaginary.[8] The Euclidean action on $\mathbb{SE}(2) \times \mathbb{C}$ is given by

(5.7) $$(x, \theta) \cdot (p, \varphi, q) = (e^{i\theta} p + x, \varphi + \theta, q), \quad \forall (x, \theta) \in \mathbb{SE}(2).$$

The analysis of (5.6), the titular *dynamical system approach*, allows the authors to recover the results of Barkley and Wulff concerning the Hopf bifurcation from a RW and resonant growth by considering a parameterized version of (5.6); a quick note on Bogdanov-Takens bifurcation from $1-$armed spirals is also provided. If $\ell > 1$, the structure of the 5 dimensional center bundle equations changes, but, as an ODE system, it retains $\mathbb{SE}(2)-$equivariance under the action

(5.8) $$(x, \theta)_m \cdot (p, \varphi, q) = (e^{i\theta} p + x, \varphi + \ell\theta, e^{im\theta} q), \quad \forall (x, \theta) \in \mathbb{SE}(2),$$

for some fixed $m \in \{0, \ldots, [\ell/2]\}$.[9]

Theorem 5.2. ([21], section 5, p. 571) *Let $\dot{y} = N(y, \mu)$ be the center bundle equations for an $\ell-$armed spiral, parameterized by $\mu \in \mathbb{R}$. There is a unique parameter value μ_0 at which the spiral undergoes a codimension-two bifurcation to resonant growth if and only if ℓ and m are coprime.*

In a subsequent paper [22], the authors again use the dynamical system approach to show that while $\mathbb{SO}(2)-$symmetry alone may explain rotating waves, Euclidean symmetry is necessary in order to observe the unbounded growth of Barkley and Wulff, as well as to explain the full bifurcation diagram of section 2.

Still, some observed spiral behaviours (see next section for a partial list) are left unexplained by a careful analysis of the center bundle equations (5.6); this obstacle is overcome through the introduction of forced Euclidean symmetry-breaking.

---

[8] The frequencies ω_{rot} and ω_{per} in (5.6) play similar roles to the parameters ω and β_0 in [4].
[9] This action is consistent with (5.7).

6. The Effects of Forced Euclidean Symmetry-Breaking

Physical experiments can never be perfectly Euclidean, if only because of their finite nature. In the heart, this reality is obvious. Cardiac tissue is *anisotropic* (*i.e.* heart fibres have a preferred orientation and electrical conductivity is direction-dependent). Furthermore, tissue distribution is not uniform: there are zones of relatively high density that affect cardiac activity [30].

Similarly, introducing light pulses in a light-sensitive BZ reaction changes the geometry of the system. Moreover, the boundary cannot be ignored when the size of the spiral core is 'comparable' to that of the domain.

Yet, the heart and the BZ reaction retain a partly Euclidean local structure. At distances 'far' from the inhomogeneities, is their effect truly felt? If the anisotropy ratio is such that the 'preferred' direction is only slightly so 'preferred', are spiral dynamics really affected? Can the Oregonator distinguish the boundary from infinity if it is 'very distant' from the spiral core?

The Euclidean model alone cannot explain these events: clearly, any model hoping to do so should incorporate *forced Euclidean symmetry-breaking* (FESB) in order to maintain the 'partly Euclidean structure' described above. The combination of Barkley's approach with FESB predicts, amongst other, the following (observed) spiral behaviours:

Spiral anchoring appears when local inhomogeneities are present: spirals are attracted or repelled by a RW which rotates around the site of the inhomogeneity. This has been observed in cardiac tissue [17] and in numerical simulations of a modified Oregonator [38].

Epicyclic drifting can be observed when the sizes of the physical domain and of the spiral core are comparable: the latter is then attracted to the boundary of the domain and rotate around it in a meandering fashion. This has been observed in experiments and numerical simulations in a light-sensitive BZ reaction [57, 59].[10]

Quasi-periodic anchoring is witnessed in periodically-forced RDS. The results are similar to spiral anchoring, with the attracting/repelling structure consisting of either two- or three-frequency quasi-periodic motion.[11] These have been observed in a light-sensitive BZ reaction which is periodically hit by light pulses and the corresponding modified Oregonator model [13, 25].

Discrete RW and MRW can be seen in systems that incorporate the notion of anisotropy (*i.e.* the system has a 'preferred direction').[12] In general, the tip path has discrete two-fold symmetry. These have been observed in numerical experiments on the bidomain model of cardiac electrophysiology [41].

Phase-locking can also be seen in systems that incorporate anisotropy: the rotation and meander frequencies can lock and this motion can be superimposed with a slow drift. This has been observed in the bidomain model as well [41].

These are illustrated in figure 6. In what follows, we present the results obtained through systematic breaking of the Euclidean symmetry.

---

[10]The term *boundary drifting* is also used.

[11]They are then called *entrainment* and *resonance attractors*, respectively.

[12]Generically, these waves cannot occur in $\mathbb{SE}(2)-$equivariant reaction-diffusion systems [7].
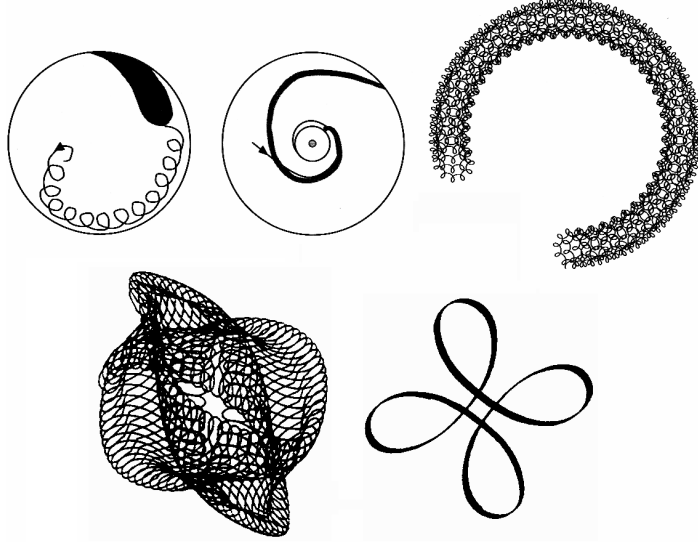
FIGURE 6. Non-standard motions of the spiral wave tip. Top row (from left to right): boundary drifting, spiral anchoring, and a three-frequency attractor [35]. Bottom row (from left to right): a discrete MRW with $\mathbb{Z}_2$−symmetry and phase-locking with small drift [41].

6.1. **A Single TSB Term.** Using the center manifold reduction theorem of [43,44], LeBlanc and Wulff [35] showed that translational symmetry-breaking (TSB) from Euclidean symmetry generically leads to anchoring or quasi-periodic anchoring and that TW, MTW, boundary drifting and quasi-periodic attractors could also occur. They did so by studying a general perturbed ODE system on the center bundle $\mathbb{SE}(2) \times \mathbb{C}$, taking the form

$$(6.1) \quad \begin{aligned} \dot{p} &= e^{i\varphi} \left[F^p(q, \overline{q}) + \varepsilon G^p(pe^{-i\varphi}, \overline{p}e^{i\varphi}, q, \overline{q}, \varepsilon) \right] \\ \dot{\varphi} &= F^\varphi(q, \overline{q}) + \varepsilon G^\varphi(pe^{-i\varphi}, \overline{p}e^{i\varphi}, q, \overline{q}, \varepsilon) \\ \dot{q} &= F^q(q, \overline{q}) + \varepsilon G^q(pe^{-i\varphi}, \overline{p}e^{i\varphi}, q, \overline{q}, \varepsilon) \end{aligned}$$

where $\varepsilon \in \mathbb{R}$ is small and the G−perturbations are bounded and uniformly continuous in p and q. When $\varepsilon = 0$, (6.1) is $\mathbb{SE}(2)$−equivariant under the action of (5.7), but for $\varepsilon \neq 0$, the system is generally only $\mathbb{SO}(2)$−equivariant: the translational symmetry of the model has been broken. The following results are proved (directly or in equivalent forms) in [35].

6.1.1. *Relative Equilibria.* In the case of normally hyperbolic (rotating) relative equilibria,[13] we may assume without loss of generality, and after an appropriate time-rescaling of the φ variable, that the center bundle equations (6.1) take the form

$$(6.2) \quad \dot{p} = e^{it} \left[v + \varepsilon H(pe^{-it}, \overline{p}e^{it}, \varepsilon) \right],$$

where $v \in \mathbb{C}^\times$ and $\varepsilon \in \mathbb{R}$ is small. Set $\widetilde{H}(w, \overline{w}, \varepsilon) = H(w - iv, \overline{w} + i\overline{v}, \varepsilon)$.

---

[13]That is, $q = 0$ in (6.1) and the RW u_* is not at the transition to a MRW.

Theorem 6.1. *Let $a = \mathrm{Re}[D_1\widetilde{H}(0,0,0)]$. If $a \neq 0$, then for all $\varepsilon \neq 0$ small enough, the center bundle equation (6.2) has a unique smooth branch of periodic solutions*

(6.3) $p_\varepsilon(t) = (-iv + O(\varepsilon))\, e^{it}, \quad \varphi(t) = t,$

whose stability is exactly determined by the sign of $a\varepsilon$.

These periodic solutions are centered around the origin in the p-plane and are observable as anchored RW in the physical space. Note that the hypotheses of theorem 6.1 are generic.

Theorem 6.2. *Let*

$$I(\rho) = \mathrm{Re}\left[\int_0^{2\pi} e^{-it}\widetilde{H}\left(\rho e^{-it}, \rho e^{it}, 0\right) dt\right].$$

If $\rho_0 > 0$ is a hyperbolic solution of $I(\rho) = 0$, then for all $\varepsilon \neq 0$ small enough, the center bundle equation (6.2) has an epicyclic solution around the origin, whose stability is exactly determined by the sign of $\varepsilon I'(\rho_0)$.

These solutions represent quasi-periodic motion around the origin in the $p-$plane and are observable as epicycle-like motion along a circular boundary in the physical space, with angular frequency given by $1 + O(\varepsilon)$. Note that the hypotheses of theorem 6.2 are not generic.

Since TW can also be seen as $\infty-$centered orbits with no rotational component, an appropriate change of variables considered with the limiting case $\omega_{\mathrm{rot}} \to 0$ takes the center bundle equations (6.1) to the equivalent system

(6.4) $\dot{z} = -vz^2 + i\varepsilon z C^\varphi(z,\overline{z},\varepsilon) - \varepsilon z^2 C^p(z,\overline{z},\varepsilon),$

where $C^\varphi(z,\overline{z},\varepsilon) = G^\varphi(z^{-1},\overline{z}^{-1},\varepsilon)$ and $C^p(z,\overline{z},\varepsilon) = G^p(z^{-1},\overline{z}^{-1},\varepsilon)$.

Theorem 6.3. *If C^φ and C^p are sufficiently smooth near $z = 0$, the center bundle equation (6.4) undergoes a transcritical bifurcation of equilibria at $\varepsilon = 0$.*

The trivial equilibria $z = 0$ represent traveling waves in the $p-$plane and are observable as linear drifts in the physical space.

6.1.2. *Relative Periodic Solutions.* In the case of normally hyperbolic (rotating) relative periodic solutions,[14] we may assume without loss of generality that, after an appropriate time-rescaling and a change of variables, the center bundle equations (6.1) take the form

(6.5)
$$\dot{w} = -i\omega_{\mathrm{rot}}w + \varepsilon H^w(w,\overline{w},t,\varepsilon)$$
$$\dot{\varphi} = \omega_{\mathrm{rot}} + \varepsilon H^\varphi(w,\overline{w},t,\varepsilon),$$

where H^w and H^φ are $2\pi-$periodic in t, $\omega_{\mathrm{rot}} \notin \mathbb{Z}$ and $w = pe^{-i\varphi}$.

Theorem 6.4. *Let $h_1^w(t) = D_w H^w(0,0,t,0)$ and set*

$$\beta = \mathrm{Re}\left[\int_0^{2\pi} h_1^w(t)dt\right].$$

If $\beta \neq 0$, then for all $\varepsilon \neq 0$ small enough, the time-2π map of the center bundle equations (6.5) has a unique smooth branch of hyperbolic fixed points w_ε whose stability is exactly determined by the sign of $\varepsilon\beta$.

---

[14]That is, the q equation in (6.1) has a $2\pi-$periodic solution and the MRW u^* is not at the transition to a RW.

These fixed points represent periodic solutions centered around the origin in the w-plane and are observable as anchored MRW in the physical space. Note that the hypotheses of theorem 6.4 are generic.

There are certain similarities between theorems 6.1 and 6.4; in the same vein, the following two results are related to theorem 6.2.

Theorem 6.5. *If $\omega_{rot} \notin \mathbb{Q}$, let*

$$J(\rho) = \mathrm{Re}\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\omega_{rot}t}H^w\left(\rho e^{-i\omega_{rot}t}, \rho e^{i\omega_{rot}t}, t, 0\right)\, dt\right].$$

If $\rho_0 > 0$ is a hyperbolic solution of $J(\rho) = 0$, then for all $\varepsilon \neq 0$ small enough, the $w-$equation in (6.5) has a unique smooth branch of hyperbolic invariant two-torii T_ε, whose stability is exactly determined by the sign of $\varepsilon J'(\rho_0)$.

Such an invariant two-torus represents an $O(\varepsilon)$ drift of a MRW (centered at a point different from the origin) around its $\mathbb{SO}(2)-$orbit about 0 in (6.5) and is observable as a three-frequency motion in the physical space. Note that the hypotheses of theorem 6.5 are not generic.

Theorem 6.6. *If $\omega_{rot} \in \mathbb{Q}$, with $\omega_{rot} = \frac{q}{\jmath^*}$, $\gcd(q, \jmath^*) = 1$ and $\jmath^* > 1$, let*

$$h_0(\xi, \overline{\xi}) = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\omega_{rot}t}H^w\left(\xi e^{-i\omega_{rot}t}, \overline{\xi}e^{i\omega_{rot}t}, t, 0\right)\, dt.$$

If $\tilde{\xi}(t)$ is a hyperbolic periodic solution [resp. an equilibrium point] of the $\mathbb{Z}_{\jmath^*}-$equivariant ODE $\dot{\xi} = h_0(\xi, \overline{\xi})$, then for all $\varepsilon \neq 0$ small enough, the $w-$equation in (6.5) has a unique smooth branch of hyperbolic invariant two-torii $\tilde{T}(\varepsilon)$ [resp. of hyperbolic $\frac{2\pi}{\jmath^*}-$periodic solutions $\tilde{w}_\varepsilon(t)$], whose stability as an invariant set is exactly determined by the product of the sign for the stability of $\tilde{\xi}(t)$ with the sign of ε.

The interpretation of these invariant two-torii and/or periodic solutions are exactly as in the remarks following theorem 6.4 and 6.5.

MTW are approached much the same way as TW were tackled in the preceding section. After an appropriate change of variables and a time-rescaling, and considering the limiting case $\omega_{\mathrm{rot}} \to 0$, the relevant equations are

$$(6.6) \qquad \begin{aligned} \dot{z} &= -vz^2 + \varepsilon E^z(z, \overline{z}, t, \varepsilon) \\ \dot{\varphi} &= \varepsilon E^\varphi(z, \overline{z}, t, \varepsilon), \end{aligned}$$

where $E^z(z, \overline{z}, t, \varepsilon) = -z^2 H^w(z^{-1}, \overline{z}^{-1}, t, \varepsilon)$, $E^\varphi(z, \overline{z}, t, \varepsilon) = H^\varphi(z^{-1}, \overline{z}^{-1}, t, \varepsilon)$ are $2\pi-$periodic in t and $v \in \mathbb{C}^\times$.

Theorem 6.7. *If E^z and E^φ are sufficiently smooth near $z = 0$, the time-2π map of the $z-$equation in the center bundle equations (6.6) undergoes a transcritical bifurcation of fixed points at $\varepsilon = 0$.*

The trivial fixed points $z = 0$ represent MTW in the $p-$plane and are observable as linear meandering in the physical space.

6.2. **A Single RSB Term.** LeBlanc [34] then showed that rotational symmetry-breaking (RSB) from Euclidean symmetry could provide an explanation for the appearance of discrete RW, discrete MRW and phase-locking in excitable media with anisotropy. He did so by studying a general perturbed ODE system on the center bundle $\mathbb{SE}(2) \times \mathbb{C}$, taking the form

$$
\begin{aligned}
\dot{p} &= e^{i\varphi}\left[F^p(q, \overline{q}) + \varepsilon G^p(\varphi, q, \overline{q}, \varepsilon)\right] \\
\dot{\varphi} &= F^\varphi(q, \overline{q}) + \varepsilon G^\varphi(\varphi, q, \overline{q}, \varepsilon) \\
\dot{q} &= F^q(q, \overline{q}) + \varepsilon G^q(\varphi, q, \overline{q}, \varepsilon)
\end{aligned}
\tag{6.7}
$$

where $\varepsilon \in \mathbb{R}$ is small, $\jmath^* \in \mathbb{N}$, and the $G-$perturbations are bounded and uniformly continuous in p and q, as well as $\frac{2\pi}{\jmath^*}-$periodic in φ. When $\varepsilon = 0$, (6.7) is $\mathbb{SE}(2)-$equivariant under the action of (5.7), but for $\varepsilon \neq 0$, the system is generally only $\mathbb{C}\dot{+}\mathbb{Z}_{\jmath^*}-$equivariant: the rotational symmetry of the model has been broken. The following results are proved (directly or in equivalent forms) in [34].

6.2.1. *Relative Equilibria.* Let $\jmath^* \geq 1$ be a fixed integer. In the case of normally hyperbolic (rotating) relative equilibria, we may assume without loss of generality that, after an appropriate time-rescaling of the φ variable, the center bundle equations (6.7) take the form

$$
\begin{aligned}
\dot{p} &= e^{i\varphi}[v + \varepsilon G^p(\varphi, \varepsilon)] \\
\dot{\varphi} &= \omega_{\text{rot}} + \varepsilon G^\varphi(\varphi, \varepsilon),
\end{aligned}
\tag{6.8}
$$

where $v \in \mathbb{C}^\times$, $\omega_{\text{rot}} \in \mathbb{R}$ and G^p, G^φ are $\frac{2\pi}{\jmath^*}-$periodic in φ.

Theorem 6.8. *Assume $\omega_{rot} \neq 0$. For all $\varepsilon \neq 0$ sufficiently small, the solutions of (6.8) are $\frac{2\pi}{\jmath^*}-$periodic in time with discrete $\mathbb{Z}_{\jmath^*}-$symmetry.*

These periodic solutions represent discrete RW in the physical space.

Theorem 6.9. *Assume $\omega_{rot} = 0$. For all $\varepsilon \neq 0$ sufficiently small, if $G^\varphi(\varphi, 0) \neq 0$ for all $\varphi \in \mathbb{S}^1$, the solutions of (6.8) are discrete RW, with (large) radii of the order of $\frac{1}{\varepsilon}$. On the other hand, if there exists $\varphi^* \in [0, 2\pi)$ such that $G^\varphi(\varphi^*, 0) = 0$ and $D_\varphi G^\varphi(\varphi^*, 0) \neq 0$, then (6.8) has at least $\jmath^*$ stable (attracting) TW solutions and an equal number of unstable (repelling) TW solutions.*

In the latter case, all solutions of (6.8) end up drifting linearly, after an initial transient period.

6.2.2. *Relative Periodic Solutions.* In the case of normally hyperbolic (rotating) relative periodic solutions,footnoteWhere the corresponding $2\pi-$periodic solution $q^*(t)$ to the q equation in (6.7) is such that $F^q(q^*(t), \overline{q^*}(t)) \neq 0$ for all t. we may assume without loss of generality that, after an appropriate time-rescaling and a change of variables, the center bundle equations (6.7) take the form

$$
\begin{aligned}
\dot{p} &= e^{i\varphi}[\tilde{F}^p(\theta) + \varepsilon \tilde{G}^p(\varphi, t, \varepsilon)] \\
\dot{\varphi} &= \omega_{\text{rot}} + \tilde{F}^\varphi(\theta) + \varepsilon \tilde{G}^\varphi(\varphi, t, \varepsilon),
\end{aligned}
\tag{6.9}
$$

where all functions are $2\pi-$periodic in θ and $\frac{2\pi}{\jmath^*}-$periodic in φ, and where the average value of $\tilde{F}^\varphi$ is 0. In the unperturbed case, all solutions of (6.9) are MTW (if $\omega_{\text{rot}} \in \mathbb{Z}$) or discrete MRW (if $\omega_{\text{rot}} \notin \mathbb{Z}$).

If $\varepsilon \neq 0$ is sufficiently small, the $\varphi-$equation in (6.9) defines a $\mathbb{Z}_{j^*}-$equivariant flow on a two-torus, with associated Poincaré map

$$(6.10) \qquad P(\varphi; \omega_{\text{rot}}, \varepsilon) = \varphi + 2\pi\omega_{\text{rot}} + \varepsilon H(\varphi, \omega_{\text{rot}}, \varepsilon),$$

where H is $\frac{2\pi}{j^*}-$periodic in φ and C^r for some $r \geq 3$. The map P is thus a circle map; denote its rotation number by $\rho_{\omega_{\text{rot}}, \varepsilon}$.

Theorem 6.10. *If $\rho_{\omega_{rot}, \varepsilon} = \frac{m}{\gamma}$, where $\gamma > 0, m \in \mathbb{Z}$ are coprime, set $k = \gcd(\gamma, j^*)$. Then, phase-locking occurs in (6.9) when $(\omega_{rot}, \varepsilon)$ lies in the $m : \gamma$ Arnol'd tongue of P.*

If $k \neq 1$, the $p-$component of solutions of (6.9) is a $\gamma-$petaled $2\pi\gamma-$periodic curve with $\mathbb{Z}_k-$spatial symmetry. Otherwise, the $p-$component of solutions of (6.9) is a superposition of a motion akin to the one in the case $k \neq 1$ together with a 'slow' linear drift.

Theorem 6.11. *If $\rho_{\omega_{rot}, \varepsilon} \notin \mathbb{Q}$, the flow on the two-torus described above is ergodic.*

If there exist $\sigma \in (0, 1)$ and $K > 0$ such that

$$\left| \rho_{\omega_{\text{rot}}, \varepsilon} + \frac{k}{j} \right| \geq K|j|^{-(2+\sigma)}$$

for all non-trivial integer pairs (k, j) (which is almost always the case), then the $p-$component of solutions of (6.9) is quasi-periodic and the closure of its positive image has $\mathbb{Z}_{j^*}-$rotational symmetry.

6.3. **Simultaneous TSB Terms.** The next logical step lies in studying the effects of n simultaneous TSB perturbations, for $n > 1$, which is done in [8, 9, 11]. Any excitable media which is littered with inhomogeneities, such as the human heart, could then in theory be modeled by a general system of ODE on the center bundle $\mathbb{SE}(2) \times \mathbb{C}$, taking the form

$$\dot{p} = e^{i\varphi} \left[F^p(q, \overline{q}) + \sum_{i=1}^{n} \lambda_i G_i^p((p - \xi_i)e^{-i\varphi}, \overline{(p - \xi_i)}e^{i\varphi}, q, \overline{q}, \lambda) \right]$$

$$(6.11) \qquad \dot{\varphi} = F^{\varphi}(q, \overline{q}) + \sum_{i=1}^{n} \lambda_i G_i^{\varphi}((p - \xi_i)e^{-i\varphi}, \overline{(p - \xi_i)}e^{i\varphi}, q, \overline{q}, \lambda)$$

$$\dot{q} = F^q(q, \overline{q}) + \sum_{i=1}^{n} \lambda_i G_i^q((p - \xi_i)e^{-i\varphi}, \overline{(p - \xi_i)}e^{i\varphi}, q, \overline{q}, \lambda)$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ is small, $\xi_1, \ldots, \xi_n \in \mathbb{C}$ are all distinct and the $G-$perturbations are bounded and uniformly continuous in p and q.

When $\lambda = 0$, (6.11) is $\mathbb{SE}(2)$-equivariant under the action of (5.7); when $\lambda \neq 0$ is near the origin and along the jth coordinate axis of $\mathbb{R}^n$, (6.11) is generally only $\mathbb{SO}(2)_{\xi_j}$-equivariant (i.e. it only commutes with rotations about the point ξ_j), and when two or more of the λ_i are not zero, the system is generally only trivially equivariant: the translational symmetry of the model has been broken.

6.3.1. *Relative Equilibria.* In the case of normally hyperbolic (rotating) relative equilibria, we may assume without loss of generality that, after an appropriate time-rescaling of the φ variable, the center bundle equations (6.11) take the form

$$(6.12) \qquad \dot{p} = e^{it} \left[v + \sum_{j=1}^{n} \lambda_j H_j((p - \xi_j)e^{-it}, (\overline{p - \xi_j})e^{it}, \lambda) \right]$$

where, $v \in \mathbb{C}^{\times}$ and the functions H_j are smooth and uniformly bounded in p. Boily, LeBlanc and Matsui study spiral anchoring in this particular setting [8, 11].

A $2\pi-$periodic solution p_λ of (6.12) is called a *perturbed rotating wave* of (6.12). Define the average value

$$[p_\lambda]_{\mathrm{A}} = \frac{1}{2\pi} \int_0^{2\pi} p_\lambda(t) \, dt.$$

If the Floquet multipliers of p_λ all lie within (resp. outside) the unit circle, we shall say that $[p_\lambda]_{\mathrm{A}}$ is the *anchoring* (resp. *repelling*, or *unstable anchoring*) *center* of p_λ.

Theorem 6.12. *Let* $k \in \{1, \ldots, n\}$ *and define* $\alpha_k = \mathrm{Re}\,[D_1 H_k(iv, -i\overline{v}, 0)]$. *If* $\alpha_k \neq 0$, *there exists a wedge-shaped region of the form*

$$\mathcal{W}_k = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : |\lambda_j| < W_{k,j}|\lambda_k|, \ W_{k,j} > 0, \ \text{for } j \neq k \text{ and } \lambda_k \text{ near } 0\}$$

such that for all $0 \neq \lambda \in \mathcal{W}_k$, *the center bundle equation* (6.12) *has a unique perturbed rotating wave* $\mathcal{S}_\lambda^k$, *with center* $[\mathcal{S}_\lambda^k]_{\mathrm{A}}$ *generically away from* ξ_k, *whose stability is uniquely determined by the sign of* $\lambda_k \alpha_k$.

In contrast to theorem 6.1, these periodic solutions are not necessarily centered around an inhomogeneity ξ_k in the $p-$plane, but they are still observable as anchored RW in the physical space. Note that the hypotheses of theorem 6.12 are generic. Furthermore, $[\mathcal{S}_\lambda^k]_{\mathrm{A}}$ is a center of anchoring when $\lambda_k \alpha_k < 0$ and a center of repelling when $\lambda_k \alpha_k > 0$.

Boily then showed that theorem 6.2 has a similar generalization [8,9]. An *epicycle manifold* of (6.12) is an invariant set $\hat{\mathcal{E}}_\lambda$ for (6.12) in which all solutions are epicycles when projected upon the $p-$plane.

Theorem 6.13. *Let* $k \in \{1, \ldots, n\}$ *and*

$$I_k(\rho) = \mathrm{Re}\left[\int_0^{2\pi} e^{-it} H_k \left(\rho e^{-it} - iv, \rho e^{it} + i\overline{v}, 0\right) dt \right].$$

If $\rho_k > 0$ *is a hyperbolic solution of* $I_k(\rho) = 0$, *then there exists a wedge-shaped region of the form*

$$\mathcal{V}_k = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : |\lambda_j| < V_{k,j}|\lambda_k|, \ V_{k,j} > 0, \ \text{for } j \neq k \text{ and } \lambda_k \text{ near } 0\}$$

such that for all $0 \neq \lambda \in \mathcal{V}_k$, (6.12) *has an epicycle manifold* $\mathcal{E}_\lambda^k$ *whose stability is exactly determined by the sign of* $\lambda_k I_k'(\rho_k)$.

These solutions represent quasi-periodic motion centered near (but not generically at) the inhomogeneity ξ_k in the $p-$plane and are observable as epicycle-like motion along a circular boundary in the physical space. Note that as was previously the case, the hypotheses of theorem 6.13 are not generic.

6.4. **Combined TSB and RSB Terms.** Yet another way in which the Euclidean symmetry can be broken lies in the combination of TSB and RSB terms; such a situation is analyzed in [8, 10]. Anisotropic media near an inhomogeneity, such as cardiac tissue in the neighbourhood of site of higher density at the origin, could then in theory be modeled by a general system of ODE on the center bundle $\mathbb{SE}(2) \times \mathbb{C}$, taking the form

$$\dot{p} = e^{i\varphi} \left[F^p(q,\overline{q}) + \varepsilon G^p(\varphi,q,\overline{q},\varepsilon,\mu) + \mu H^p(pe^{-i\varphi},\overline{p}e^{i\varphi},q,\overline{q},\varepsilon,\mu) \right]$$

(6.13) $$\dot{\varphi} = F^\varphi(q,\overline{q}) + \varepsilon G^\varphi(\varphi,q,\overline{q},\varepsilon,\mu) + \mu H^\varphi(pe^{-i\varphi},\overline{p}e^{i\varphi},q,\overline{q},\varepsilon,\mu)$$

$$\dot{q} = F^q(q,\overline{q}) + \varepsilon G^q(\varphi,q,\overline{q},\varepsilon,\mu) + \mu H^q(pe^{-i\varphi},\overline{p}e^{i\varphi},q,\overline{q},\varepsilon,\mu),$$

where $(\varepsilon,\mu) \in \mathbb{R}^2$ is small, $\jmath^* \in \mathbb{N}$, the G, H−perturbations are bounded and uniformly continuous in p and q, and the G−perturbations are $\frac{2\pi}{\jmath^*}$−periodic in φ. Throughout this section, we fix $\jmath^* \in \mathbb{N}$.

Let $\mathbb{C}\dotplus\mathbb{Z}_{\jmath^*}$ be the subgroup of $\mathbb{SE}(2)$ containing all translations and rotations by angle $\frac{2\pi k}{\jmath^*}$, $k \in \mathbb{Z}$. When $(\varepsilon,\mu) = 0$, (6.13) is $\mathbb{SE}(2)$-equivariant under the action of (5.7); when $\varepsilon \neq 0$ is small and $\mu = 0$, it is $\mathbb{C}\dotplus\mathbb{Z}_{\jmath^*}$−equivariant; when $\mu \neq 0$ is small and $\varepsilon = 0$, it is $\mathbb{SO}(2)$−equivariant, and it is generally only trivially equivariant otherwise: both the translational symmetry and the rotational symmetry of the model has been broken.

In the case of normally hyperbolic (rotating) relative equilibria, we may assume without loss of generality that, after an appropriate time-rescaling of the φ variable, the center bundle equations (6.11) take the form

(6.14) $$\dot{p} = e^{it}\left[v + \varepsilon G(t,\varepsilon,\mu) + \mu H(pe^{-it},\overline{p}e^{it},\varepsilon,\mu) \right],$$

where $(\varepsilon,\mu) \in \mathbb{R}^2$, $v \in \mathbb{C}$, G and H are smooth and uniformly bounded in p, and G is $\frac{2\pi}{\jmath^*}$−periodic in t. The following results are proved (directly or in equivalent forms) in [8, 10]; they depend greatly on the nature of $\jmath^*$.

6.4.1. *The Case $\jmath^* = 1$.* As G is then 2π−periodic in t, it can be written as the uniformly convergent Fourier series

(6.15) $$G(t,\varepsilon,\mu) = \sum_{n\in\mathbb{Z}} g_n(\varepsilon,\mu)e^{int}.$$

Theorem 6.14. *If $g_{-1}(0,0) \neq 0$ and if $c_1 = \mathrm{Re}[D_1 H(-iv, i\overline{v}, 0, 0)] \neq 0$, there exists a wedge-shaped region of the form*

$$\mathcal{W} = \{(\varepsilon,\mu) \in \mathbb{R}^2 \,:\, |\varepsilon| < K|\mu|, \ \ K > 0, \ \mu \ near\ 0\,\}$$

such that for all $(\varepsilon,\mu) \in \mathcal{W}$ with $\varepsilon \neq 0$, the center bundle equation (6.14) has a unique hyperbolic discrete rotating wave $\mathcal{D}^1_{\varepsilon,\mu}$ with trivial spatio-temporal symmetry, centered near but generically not at the origin, whose stability is exactly determined by the sign of μc_1.

Theorem 6.15. *Let*

$$R(\rho) = \mathrm{Re}\left[\int_0^{2\pi} e^{-it} H\left(\rho e^{-it} - iv, \rho e^{it} + iv, 0, 0 \right) dt \right].$$

If $\rho_0 > 0$ is a hyperbolic solution of $R(\rho) = 0$, then there exists a wedge-shaped region of the form

$$\mathcal{V} = \{(\varepsilon,\mu) \in \mathbb{R}^2 \,:\, |\varepsilon| < K|\mu|, \ \ K > 0, \ \mu\ near\ 0\,\}$$

such that for all $(\varepsilon, \mu) \in \mathcal{V}$ with $\varepsilon \neq 0$, the center bundle equation (6.14) has an epicycle manifold $\mathcal{G}^1_{\varepsilon,\mu}$ centered near but generically not at the origin, whose stability is exactly determined by the sign of $\mu R'(\rho_0)$.

To get the complete picture (in both of these theorems), the situation would also need to be analyzed near the ε−axis. This implies dealing with fixed points of maps at ∞; as of now, it has been relegated to a future investigation. Note that the hypotheses of theorem 6.14 are generic, while those of 6.15 are not.

6.4.2. *The Case $\jmath^* > 1$.* In fully anisotropic media, one would have $\jmath^* = 2$.

Theorem 6.16. *Let c_1 be as in theorem 6.14. If $c_1 \neq 0$, then there exists a small deleted neighbourhood $\mathcal{W}^{\jmath^*}$ of the origin, such that for all $0 \neq (\varepsilon, \mu) \in \mathcal{W}^{\jmath^*}$, the center bundle equation (6.14) has a unique hyperbolic discrete rotating wave $\mathcal{D}^{\jmath^*}_{\varepsilon,\mu}$ with $\mathbb{Z}_{\jmath^*}$−spatio-temporal symmetry, centered at the origin in the p−plane, whose stability is exactly determined by the sign of μc_1.*

Theorem 6.17. *Let $R(\rho)$ be as in theorem 6.15. If $\rho_0 > 0$ is a hyperbolic solution of $R(\rho) = 0$, then there exists a small deleted neighbourhood $\mathcal{V}^{\jmath^*}$ of the origin, such that for all $(\varepsilon, \mu) \in \mathcal{V}^{\jmath^*}$, the center bundle equation (6.14) has an epicycle manifold $\mathcal{G}^{\jmath^*}_{\varepsilon,\mu}$ centered at the origin, whose stability is exactly determined by the sign of μc_1.*

Contrary to what might be thought at first, the epicycle manifolds of theorem 6.17 do *not* have $\mathbb{Z}_{\jmath^*}$−spatio-temporal symmetry; however, the epicycles themselves possess this symmetry in an appropriate frame of reference. As has been the case throughout this article, the hypotheses of theorem 6.16 are generic, while those of theorem 6.17 are not.

7. Conjectures, Related Results and Future Work

The application of FESB to the study of rotating waves has yielded a number of interesting verifiable results: the predictive power of the dynamical approach cannot be denied. Yet the full picture of spiral wave dynamics is far from complete

7.1. **Conjectures.** In this section, we present some conjectures concerning modulated rotating waves.

7.1.1. *Simultaneous TSB Terms.* The equations describing the essential dynamics of a normally hyperbolic modulated rotating wave are similar to those of rotating waves. Near such a MRW, the center bundle equations (6.7) are equivalent to

$$(7.1) \quad \begin{aligned} \dot{p} &= e^{i\varphi}\left[v + \sum_{j=1}^{n} \lambda_j H_j^p((p - \xi_j)e^{-i\varphi}, \overline{(p - \xi_j)}e^{i\varphi}, t, \lambda)\right] \\ \dot{\varphi} &= \omega_{\mathrm{rot}} + \sum_{j=1}^{n} \lambda_j H_j^\varphi((p - \xi_j)e^{-i\varphi}, \overline{(p - \xi_j)}e^{i\varphi}, t, \lambda), \end{aligned}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $v \in \mathbb{C}$, $\omega_{\mathrm{rot}} \neq 0$ and the functions $H_j^{p,\varphi}$ are smooth, uniformly bounded in p for all $j = 1, \ldots, n$ and 2π−periodic in t.

System (7.1) cannot be analyzed as easily as (6.12), but it seems nonetheless likely that the following hold.

Conjecture 7.1. *Let $k \in \{1, \ldots, n\}$. Generically, there is a wedge-shaped region of the form*

$$\mathcal{W}_k = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \, : \, |\lambda_j| < W_{k,j}|\lambda_k|, \ W_{k,j} > 0, \text{ for } j \neq k \text{ and } \lambda_k \text{ near } 0\}$$

such that for all $0 \neq \lambda \in \mathcal{W}_k$, the center bundle equations (7.1) have a unique perturbed modulated rotating wave solution $\mathfrak{S}_\lambda^k$, centered generically away from ξ_k.

Conjecture 7.2. *Let $k \in \{1, \ldots, n\}$. Given a hyperbolic equilibrium of a related averaged system, there is a wedge-shaped region of the form*

$$\mathcal{V}_k = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \, : \, |\lambda_j| < V_{k,j}|\lambda_k|, \ V_{k,j} > 0, \text{ for } j \neq k \text{ and } \lambda_k \text{ near } 0\}$$

such that for all $0 \neq \lambda \in \mathcal{V}_k$, the center bundle equations (7.1) have a hyperbolic $3-$frequency epicycle manifold $\mathfrak{E}_\lambda^k$ centered near, but generically not at, ξ_k.

7.1.2. *Combined RSB and TSB Terms.* Fix $\jmath^* \in \mathbb{N}$. The equations describing the essential dynamics of a normally hyperbolic modulated rotating wave are similar to those of rotating waves. Near such a MRW, the center bundle equations (6.7) are equivalent to

$$(7.2) \qquad \begin{aligned} \dot{p} &= e^{i\varphi} \left[v + \varepsilon G^p(\varphi, t, \varepsilon, \mu) + \mu H^p(pe^{-i\varphi}, \overline{p}e^{i\varphi}, t, \varepsilon, \mu) \right] \\ \dot{\varphi} &= \omega_{\text{rot}} + \varepsilon G^\varphi(\varphi, t, \varepsilon, \mu) + \mu H^\varphi(pe^{-i\varphi}, \overline{p}e^{i\varphi}, t, \varepsilon, \mu), \end{aligned}$$

where $(\varepsilon, \mu) \in \mathbb{R}^2$, $v \in \mathbb{C}$, $\omega_{\text{rot}} \neq 0$, the functions $G^{p,\varphi}, H^{p,\varphi}$ are smooth and uniformly bounded in p and $2\pi-$periodic in t, and the functions $G^{p,\varphi}$ are additionally $\frac{2\pi}{\jmath^*}-$periodic in φ.

System (7.2) cannot be analyzed as easily as (6.14), it seems nonetheless likely that the following hold.

Conjecture 7.3. *Let $\jmath^* = 1$. Generically, there is a wedge-shaped region of the form*

$$\mathcal{W} = \{(\varepsilon, \mu) \in \mathbb{R}^2 \, : \, |\varepsilon| < K|\mu|, \ K > 0, \ \mu \text{ near } 0\}$$

such that for all $(\varepsilon, \mu) \in \mathcal{W}$ with $\varepsilon \neq 0$, the center bundle equations (7.2) have a unique hyperbolic discrete modulated rotating wave $\mathfrak{D}_{\varepsilon,\mu}^1$, with trivial spatio-temporal symmetry, centered away from the origin.

Conjecture 7.4. *Let $\jmath^* = 1$. Given a hyperbolic equilibrium of a related averaged system, there is a wedge-shaped region of the form*

$$\mathcal{V} = \{(\varepsilon, \mu) \in \mathbb{R}^2 \, : \, |\varepsilon| < K|\mu|, \ K > 0, \ \mu \text{ near } 0\}$$

such that for all $(\varepsilon, \mu) \in \mathcal{V}$ with $\varepsilon \neq 0$, the center bundle equations (7.2) have a hyperbolic $3-$frequency epicycle manifold $\mathfrak{E}_{\varepsilon,\mu}^1$ centered near but generically not at the origin.

The remark following theorem 6.15 is likely to hold for theorems 7.3 and 7.4.

Conjecture 7.5. *Let $\jmath^* > 1$. Generically, there is a deleted neighbourhood $\mathcal{W}^{\jmath^*}$ of the origin such that such that for all $(\varepsilon, \mu) \in \mathcal{W}^{\jmath^*}$, (7.2) has a unique hyperbolic discrete modulated rotating wave $\mathfrak{D}_{\varepsilon,\mu}^{\jmath^*}$ with $\mathbb{Z}_{\jmath^*}-$spatio-temporal symmetry centered at the origin.*

Conjecture 7.6. *Let $\jmath^* > 1$. Given a hyperbolic equilibrium of a related averaged system, there is a deleted neighbourhood $\mathcal{V}^{\jmath^*}$ of the origin such that for all $(\varepsilon, \mu) \in \mathcal{V}^{\jmath^*}$, (7.2) has a hyperbolic $3-$frequency epicycle manifold $\mathfrak{E}_{\varepsilon,\mu}^{\jmath^*}$ centered at the origin.*

7.2. **Remarks.** While numerical experiments show without the shadow of a doubt that spirals can anchor away from a center of inhomogeneity [8, 10, 11], this writer would find it very satisfying to see this result reproduced in the laboratory. From a resolutely profane perspective, the Belousov-Zhabotinsky reaction appears most likely to yield results, as it seems the easiest to control.

It should also be noted that the study of spirals does not start and end in the plane. For instance, Comanici used the dynamical system approach in her doctoral thesis to study spirals on spherical domains [16]. Scroll waves, the 3-dimensional analogues of spiral waves, have also attracted attention from the physics and cardiology communities in recent years [1, 15, 47, 48].

7.3. **Future and Related Work.** We finish this paper with a list of problems and open questions: their solutions would improve our knowledge and understanding of spiral waves and excitable media.

7.3.1. *MRW, TW and MTW.* When the FESB becomes too complex, only RW and TW are easily amenable to characterization. Even then, TW have not yet been tackled. The current approach [9–11] needs to be suitably modified so as to accommodate the extra variable that appears in (7.1) and (7.2).

7.3.2. *An Explicit Center Manifold Reduction Theorem.* While extremely powerful, the CMRT has the disadvantage of being a strict existence theorem: it tells us that the dynamics on the center manifold are given by an ODE system with certain symmetries, but it does not provide the explicit relation between that system and the original semi-flow.

In particular, when a specific center manifold system is studied, we have no way of knowing if it corresponds to a 'viable' excitable system, *i.e.* if it is 'attainable' in any way from such an excitable system *via* the CMRT.

Recent observations by Lajoie and LeBlanc [33] suggest that it might be possible to efficiently relate the coefficients of a RDS to those of the center manifold near a traveling wave. Is there a direct and efficient way to compute the relevant coefficients of the center bundle equations directly from the PDE, near any type of relative equilibrium or relative periodic solution?

7.3.3. *Spiral Groupings.* So far, the model-independent approach based on the CMRT has only been used to study *isolated* spiral waves. Even though experiments by Li, Ouyang, Petrov and Swinney [36] have shown that spiral waves can be isolated with the help of a laser, they are rarely found in that state in excitable media (see for instance the two illustrations on p. 2).

Spiral groupings, where two or more spirals rotate around a common center or one another, have much different dynamics, as can be attested by the recent numerical simulations of Pertsov and Zariski [58]. Some of the showcased interactions are somewhat analogous to already-obtained results about epicycle drifting, which begs the question: how can the current approach be altered to apply to spiral groupings as well?

7.3.4. *Global Spiral Dynamics.* Finally, the CMRT can only be applied to local neighbourhoods of spiral wave solutions. Yet, many spiral interactions are global in nature (see [58] for details). As of now, there is little machinery short of numerical simulations to deal with spiral dynamics on a global level. How can this situation be remedied?

Appendix A. Selected Bibliography of Spiral Pattern Formation

BZ Reaction and the Oregonator − $[25, 36, 38, 59]$.
Cardiac Tissue and the FHN Equations − $[17, 29, 37, 41, 52, 55, 57]$.
Global Spiral Dynamics − $[26, 58]$.
Surveys − $[28, 53]$.
Dynamical System Approach − $[4, 6\text{–}11, 16, 21, 22, 34, 35, 44, 45, 56]$.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF OTTAWA, OTTAWA, ONTARIO, K1N 6N5, CANADA
  E-mail address: `pboily@uottawa.ca`